

Massive spin-2 and spin- $\frac{1}{2}$ no hair theorems for stationary axisymmetric black holes

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Abstract

We present a proof of the no hair theorems corresponding to free massive non-perturbative Pauli-Fierz spin-2 and perturbative massive spin- $\frac{1}{2}$ fields for stationary axisymmetric de Sitter black hole spacetimes of dimension four with two commuting Killing vector fields. The applicability of these results for asymptotically flat and anti-de Sitter spacetimes are also discussed.

Keywords: Stationary axisymmetric black holes, no hair theorem, spinor, de Sitter

1 Introduction

The classical black hole no hair conjecture states that any realistic gravitational collapse reaches a final stationary state characterized by a small number of parameters. A part of this conjecture has been proven mathematically rigorously by taking different matter fields, known as the no hair theorem, (see e.g. [1, 2, 3, 4] and references therein) and deals with the uniqueness of stationary black hole solutions characterized only by mass, angular momentum, and charges corresponding only to long range gauge fields. If a stationary black hole spacetime supports in its exterior any non-trivial field configuration other than long range gauge fields, the former one is called as ‘hair’. Thus proving no hair theorems means to show that there cannot exist any non-trivial and physically reasonable field configuration other than long range gauge fields in the exterior of the black hole

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spacetime. In particular, it has been shown that static, spherically symmetric black hole spacetimes do not support hair corresponding to scalars in convex potentials, Proca-massive vector field [5], or even gauge fields corresponding to the Abelian Higgs model [6, 7].

However, all the above proofs assume asymptotic flatness, i.e. one can reach spatial infinity and sufficiently rapid fall-off conditions can be imposed upon the matter fields there. But recent observations suggest that there is a strong possibility that our universe is dominated by some exotic matter exerting negative pressure such as a positive cosmological constant Λ [8, 9]. It is expected in that case that the spacetime in its stationary state would possess an outer or cosmological Killing horizon [10]. For known and exact stationary solutions with a positive Λ [11], the cosmological Killing horizon acts in general as a causal boundary (see e.g. [12]) so that no observer can communicate with region beyond this horizon along a future directed path. If there is a black hole, the black hole event horizon will be located inside the cosmological horizon and the spacetime is then known as a de Sitter black hole spacetime. The observed value of Λ is tiny, of the order of 10^{-52}m^{-2} , and for such a small value the known solutions show that the cosmological horizon has a length scale $\sim \mathcal{O}(\Lambda^{-\frac{1}{2}})$. This is of course large, but not infinite. Since no physical observer can communicate beyond the cosmological horizon, in a de Sitter black hole spacetime the cosmological horizon serves as a natural boundary along with the black hole horizon. So in general one cannot impose any precise asymptotic fall off for the matter fields in the vicinity of the cosmological horizon, nor can set $T_{ab} = 0$ there. Therefore, the generalization of the no hair theorems for de Sitter black holes are expected to be different from the $\Lambda \leq 0$ cases.

In fact considerable progress has been made in this topic for static de Sitter black holes. Price's theorem, a perturbative no hair theorem [13], was proved in [14] for massless perturbations in the Schwarzschild-de Sitter background. Later the non-perturbative black hole no hair theorems were extended for a general static de Sitter black hole spacetime in [15]. Notably a violation of the standard no hair theorem was found – a spherically symmetric electrically charged solution sitting on the false vacuum of the complex scalar of the Abelian Higgs model was obtained which has no $\Lambda \leq 0$ analogue. In fact this charged solution suggests that even though Λ is tiny, the existence of the cosmological horizon as an outer boundary of the spacetime, because of the non-trivial boundary conditions, may change local physics considerably. For some more aspects on no hair theorems in such spacetimes we refer our reader to [16, 17].

So it is an interesting task to generalize the no hair theorems for stationary de Sitter black holes. For an asymptotically flat spacetime, the no hair proofs for a rotating black hole for scalar and Proca fields were given in [18]. The $\Lambda > 0$ coordinate independent generalization of these proofs can be found in [19]. For a discussion on the (2+1)-dimensional no hair theorem see [20]. See also [21] for a scalar no hair theorem in stationary axisymmetric asymptotically flat spacetimes with non-minimal matter-gravity coupling.

In this paper we shall give a proof of the classical no hair theorems corresponding to massive Pauli-Fierz spin-2 [22] and spin- $\frac{1}{2}$ fields for stationary axisymmetric de Sitter black hole spacetimes. For static asymptotically flat spherically symmetric spacetime, a proof of spin-2 no hair can be found in [18]. It was shown later by constructing Wu-Yang's magnetic monopole in such spacetimes that although classical spin-2 hair is ruled out, quantum hair is not, which can be detected via a stringy generalization of the Bohm-Aharonov effect [23, 24]. We shall also address briefly this phenomenon for these spacetimes. It was shown in [14] that the Schwarzschild-de Sitter spacetime does not support massless $\text{SL}(2, \mathbb{C})$ spinor hair with vanishing frequency. For demonstration of the spin- $\frac{1}{2}$ no hair theorem via time dependent perturbation technique we refer our reader to [25, 26, 27]. We further refer our reader to e.g. [28, 29, 30, 31] and references therein for recent developments including observational aspects of the no hair theorem.

The paper is organized as follows. In the next section we outline all the necessary assumptions and the geometrical set up we work in. In Sec.s 3 and 4 we give respectively the proofs of the classical no hair theorems for the massive spin-2 and spin- $\frac{1}{2}$ fields. Finally we discuss our results.

We shall set $c = G = \hbar = 1$ throughout. We shall take mostly negative signature $(+, -, -, -)$

for the spacetime metric. For an orthonormal basis $e_{(a)}^b$, the index in parenthesis will always correspond to local Lorentz frame.

2 Assumptions and the geometrical set up

In the following we outline the assumptions and the geometrical set up of the spacetime we work in, details of which can be found in [10].

The spacetime is a (3+1)-dimensional, smooth, connected, orientable, Hausdorff and para-compact stationary axisymmetric manifold with a Lorentzian metric g_{ab} , admits a spin structure, satisfies Einstein's equations and is endowed with two commuting Killing vector fields $\{\xi^a, \phi^a\}$,

$$\nabla_{(a}\xi_{b)} = 0 = \nabla_{(a}\phi_{b)}, \quad (1)$$

$$[\xi, \phi]^a = \mathcal{L}_\xi \phi^a = \xi^b \nabla_b \phi^a - \phi^b \nabla_b \xi^a = 0. \quad (2)$$

ξ^a is locally timelike with norm $\xi^a \xi_a = +\lambda^2$ and generates the stationarity, whereas ϕ^a is locally spacelike with closed orbits with parameter $0 \leq \phi \leq 2\pi$ and norm $\phi^a \phi_a = -f^2$ and hence generates the axisymmetry. We assume that the spacetime connection '∇' is torsion free, i.e. for any at least twice differentiable spacetime function $\varepsilon(x)$ we have identically,

$$\nabla_{[a}\nabla_{b]}\varepsilon(x) = 0. \quad (3)$$

A basis for this spacetime can be chosen as $\{\xi^a, \phi^a, \mu^a, \nu^a\}$, where $\{\mu^a, \nu^a\}$ are spacelike basis vectors orthogonal to both ξ^a and ϕ^a . We assume that the spacelike 2-'planes' spanned by $\{\mu^a, \nu^a\}$ form integral submanifolds, i.e. μ^a and ν^a form the basis of a Lie algebra.

For a stationary axisymmetric spacetime in general $\xi^a \phi_a \neq 0$, so the basis $\{\xi^a, \phi^a, \mu^a, \nu^a\}$ is not orthogonal. Thus unlike static spacetimes, there exists no family of spacelike hypersurfaces which is both tangent to ϕ^a and orthogonal to ξ^a . Let us then first construct a family of spacelike hypersurfaces tangent to ϕ^a , which will be convenient for our calculations. Let us define χ_a as

$$\chi_a = \xi_a + \frac{1}{f^2} (\xi_b \phi^b) \phi_a \equiv \xi_a + \alpha \phi_a, \quad (4)$$

so that we have $\chi_a \phi^a = 0$ everywhere. Also,

$$\chi_a \chi^a = (\lambda^2 + \alpha^2 f^2) = \beta^2, \quad (5)$$

so that χ_a is timelike when $\beta^2 > 0$. The basis $\{\chi^a, \phi^a, \mu^a, \nu^a\}$ thus serves as an orthogonal basis for the spacetime. However, we note that χ^a is not a Killing field

$$\nabla_{(a}\chi_{b)} = \phi_a \nabla_b \alpha + \phi_b \nabla_a \alpha. \quad (6)$$

We also note the following vanishing Lie derivatives which follow immediately from Eq.s (1) and (2),

$$\mathcal{L}_\chi \beta = 0 = \mathcal{L}_\phi \beta, \quad \mathcal{L}_\chi \alpha = 0 = \mathcal{L}_\phi \alpha, \quad \mathcal{L}_\chi f = 0 = \mathcal{L}_\phi f. \quad (7)$$

In other words the 1-forms $\nabla_a \beta$, $\nabla_a \alpha$ and $\nabla_a f$ are all orthogonal to both χ^a and ϕ^a . This will prove useful later.

Our assumption that $\{\mu^a, \nu^a\}$ span an integral 2-submanifold and Eq.s (7) imply that χ^a satisfies the Frobenius condition of hypersurface orthogonality [10],

$$\chi_{[a} \nabla_b \chi_{c]} = 0. \quad (8)$$

Thus χ^a is orthogonal to the spacelike $\{\phi^a, \mu^a, \nu^a\}$ hypersurfaces, say Σ . Using Eq.s (6) and (8), we get an useful expression

$$\nabla_a \chi_b = \beta^{-1} \chi_{[b} \nabla_{a]} \beta + \frac{1}{2} \phi_{(a} \nabla_{b)} \alpha. \quad (9)$$

We are dealing with a stationary axisymmetric spacetime with two Killing horizons. One is the black hole horizon and the larger one which surrounds the black hole is the cosmological horizon. Let us now locate the horizons in terms of the orthogonal basis $\{\chi^a, \phi^a, \mu^a, \nu^a\}$. A stationary axisymmetric spacetime with a black hole is in general rotating and in that case ξ^a becomes spacelike within the ergosphere [36], so for such spacetimes the surface $\lambda^2 = 0$ does not in general define a horizon. It was shown in [10] by considering the null geodesic congruence tangent to a ‘closed’ $\beta^2 = 0$ hypersurface \mathcal{H} that the function α is a constant on \mathcal{H} and the orthogonal vector field χ^a coincides with a null Killing field there. Thus any such surface \mathcal{H} is essentially a Killing or true horizon. Accordingly, we define the black hole and the cosmological event horizons to be the two ‘closed’ $\beta^2 = 0$ surfaces, the former being located inside the second, such that χ^a is timelike in the region between them, becoming null on the surfaces. An example of this is the Kerr-Newman-de Sitter family of spacetimes [12].

We note that there could be a Cauchy horizon too, located inside the black hole event horizon. This is another closed $\beta^2 = 0$ surface on which χ^a is Killing and null, however the vector field χ^a is spacelike between this surface and the event horizon. The existence of the Cauchy horizon makes the black hole singularity timelike, resulting in interesting consequences in analytically extended charts [32]. Perturbative studies show that the Cauchy horizon can be unstable. We refer our readers to [32] (also references therein) for an excellent account on this for $\Lambda = 0$. For $\Lambda > 0$, this result was generalized later in [33]. We further refer our reader to [12] for maximal analytic extension of the Kerr-Newman-de Sitter spacetime including the Cauchy horizon. However it is sufficient for our present purpose to consider only the region between the black hole event horizon and the cosmological horizon, and we can safely ignore the inner Cauchy horizon if it exists.

For convenience of our calculation, we shall specify μ^a now. On any of \mathcal{H} we know that

$$\nabla_a \beta^2 = 2\kappa \chi_a, \quad (10)$$

where κ is a constant on \mathcal{H} known as the respective surface gravity. Keeping in mind that $\nabla_a \beta^2$ is orthogonal to both χ^a and ϕ^a (Eq.s (7)), we define

$$\mu_a := \frac{1}{2\kappa(x)} \nabla_a \beta^2, \quad (11)$$

where $\kappa(x)$ is a function which smoothly reaches κ when we reach \mathcal{H} . With this choice μ^a is itself Killing and null on \mathcal{H} and vanishes there as $\mathcal{O}(\beta^2)$. When the black hole is extremal, i.e. $\kappa = 0$, we simply write $\mu^a = \frac{1}{2} \nabla_a \beta^2$.

The projector $h_a{}^b$ which projects tensors onto the spacelike hypersurfaces Σ is defined as

$$h_a{}^b = \delta_a{}^b - \beta^{-2} \chi_a \chi^b. \quad (12)$$

Let D_a be the spacelike induced connection defined via the projector as $D_a \equiv h_a{}^b \nabla_b$. Then we can project the derivative of a tensor $T_{a_1 a_2 \dots}{}^{b_1 b_2 \dots}$ onto Σ as

$$D_a \tilde{T}_{a_1 a_2 \dots}{}^{b_1 b_2 \dots} := h_a{}^b h_{a_1}{}^{c_1} \dots h^{b_1}{}_{d_1} \dots \nabla_b T_{c_1 c_2 \dots}{}^{d_1 d_2 \dots}, \quad (13)$$

where \tilde{T} is the projection of T onto Σ , given by $\tilde{T}_{a_1 a_2 \dots}{}^{b_1 b_2 \dots} := h_{a_1}{}^{c_1} \dots h^{b_1}{}_{d_1} \dots T_{c_1 c_2 \dots}{}^{d_1 d_2 \dots}$. It is easy to verify that the induced connection D_a on Σ defined in Eq. (13) satisfies the Leibniz rule and is compatible with the induced metric h_{ab} . For our purpose we shall also need to act ‘ D ’ on a full spacetime tensor T by

$$D_a T_{a_1 a_2 \dots}{}^{b_1 b_2 \dots} := h_a{}^b \nabla_b T_{a_1 a_2 \dots}{}^{b_1 b_2 \dots}, \quad (14)$$

in which it is clear that D_a is merely the spacelike directional derivative associated with the full metric.

We shall also need to project tensors onto the integral 2-planes orthogonal to both χ^a and ϕ^a and spanned by μ^a and ν^a , say $\overline{\Sigma}$. The projection tensor is given by

$$\pi_a{}^b = \delta_a{}^b - \beta^{-2} \chi_a \chi^b + f^{-2} \phi_a \phi^b. \quad (15)$$

The projected derivative ' \overline{D} ' on $\overline{\Sigma}$ can be defined exactly in the same way as above.

Using the fact that the 2-planes spanned by μ^a and ν^a are integral submanifolds, we can derive the following expression for the derivative of the Killing field ϕ^a [10],

$$\nabla_a \phi_b = f^{-1} \phi_{[b} \nabla_{a]} f + \frac{1}{2} \chi_{[a} \nabla_{b]} \alpha. \quad (16)$$

We assume that there is no naked curvature singularity anywhere in our region of interest, i.e. anywhere between the two horizons including both of them. The Einstein equation $G_{ab} + \Lambda g_{ab} = 8\pi T_{ab}$ then implies that the invariants constructed from the energy-momentum tensor T_{ab} are bounded everywhere in our region of interest.

We assume that any physical matter field, or any observable concerning the matter field also obeys the symmetries of the spacetime, be it continuous or discrete, because otherwise the matter field may itself break those symmetries. In other words, if X is a physical matter field or a component of it, or an observable quantity associated with it, we must have

$$\mathcal{L}_\xi X = 0 = \mathcal{L}_\phi X. \quad (17)$$

Apart from the existence of the cosmological horizon as an outer boundary and regularity, no asymptotics on spacetime or matter fields will be imposed. However unlike the spin-2 field, we shall ignore backreaction of the spinor on the spacetime since spinors do not obey any classical energy condition [34]. We shall not consider any coupling of the spinor with gauge fields. We shall not explicitly solve Einstein's equations but shall only examine the existence of solutions of matter fields.

Being equipped with all this, we are now ready to go into the no hair proofs.

3 Massive spin-2 field

Let us begin with the massive and real spin-2 field M_{ab} . An equation of motion for M_{ab} can be written as [18, 22]

$$\nabla_c \nabla^c \left(M_{ab} - \frac{1}{2} M g_{ab} \right) + m^2 \left(M_{ab} - \frac{1}{2} M g_{ab} \right) = 0. \quad (18)$$

M_{ab} is symmetric in its two indices, $M = M_{ab} g^{ab}$ and m can be interpreted as the rest mass of the field. M_{ab} satisfies the condition : $\nabla_a M^a{}_b = 0$. We note here that unlike the gravitational perturbation equation, a pure spin-2 field theory has some ambiguities in its coupling with spacetime curvature. In particular, Eq. (18) might have contained terms like $R_{acbd} M^{cd}$. However, under the reasonable assumption that the Compton wavelength of the field is small compared to the size of the black hole horizon, the mass term always dominates over such terms outside the horizon [18]. So, we shall not consider non-minimal coupling of the field with curvature.

We take the trace of Eq. (18) and note that since M is a scalar, $\mathcal{L}_\chi M = \mathcal{L}_\xi M + \alpha \mathcal{L}_\phi M = 0$. Using this and Eq. (12), we find

$$\nabla_a \nabla^a M = \frac{1}{\beta h} \partial_c [\beta h g^{cd} \partial_d M] = \frac{1}{\beta h} \partial_c [\beta h h^{cd} \partial_d M] = \frac{1}{\beta} D_a (\beta D^a M), \quad (19)$$

where h is the determinant of the induced metric h_{ab} . Thus the trace of Eq. (18) is equivalent to

$$D_a (\beta D^a M) + m^2 \beta M = 0, \quad (20)$$

which we multiply with M and integrate by parts on Σ between the two horizons. The total divergence term is converted to a surface integral on \mathcal{H} ($\beta = 0$) and goes away leaving with us the vanishing volume integral,

$$\int_{\Sigma} \beta [-(D_a M)(D^a M) + m^2 M^2] = 0, \quad (21)$$

which shows $M = 0$ throughout.

In four spacetime dimensions M_{ab} has ten components,

$$M_{ab} = \Psi^{(1)}\chi_a\chi_b + \Psi^{(2)}\phi_a\phi_b + \Psi^{(3)}\mu_a\mu_b + \Psi^{(4)}\nu_a\nu_b + \Psi^{(5)}\chi_{(a}\phi_{b)} + \Psi^{(6)}\chi_{(a}\mu_{b)} + \Psi^{(7)}\chi_{(a}\nu_{b)} + \Psi^{(8)}\phi_{(a}\mu_{b)} + \Psi^{(9)}\phi_{(a}\nu_{b)} + \Psi^{(10)}\mu_{(a}\nu_{b)}, \quad (22)$$

where $\Psi^{(i)}$'s are scalars. To simplify our calculations, we shall now use the discrete symmetry of the spacetime to get rid of some of these components of M_{ab} . The metric for a stationary axisymmetric spacetime under consideration is invariant under the simultaneous reflections $\xi_a \rightarrow -\xi_a$ and $\phi_a \rightarrow -\phi_a$. Eq. (4) then shows these are equivalent to $\chi_a \rightarrow -\chi_a$ and $\phi_a \rightarrow -\phi_a$. Since we are not ignoring backreaction, any physical matter field must obey these symmetries [18, 35]. Noting that all the scalars in Eq. (22) are independent of parameters along ξ^a and ϕ^a , we find that the invariance under the discrete symmetry implies $\Psi^{(6)} = \Psi^{(7)} = \Psi^{(8)} = \Psi^{(9)} = 0$.

Thus we are left with six components of $M_{ab} : \{\chi\chi, \phi\phi, \chi\phi, \mu\mu, \nu\nu, \mu\nu\}$. For simplicity of notation, we shall denote the orthogonal directions (χ, ϕ, μ, ν) as (0, 1, 2, 3) respectively.

Since M_{ab} is a physical matter field, by Eq. (17) we have $\mathcal{L}_{\xi}M_{ab} = 0 = \mathcal{L}_{\phi}M_{ab}$. This gives

$$\mathcal{L}_{\chi}M_{ab} = \chi^c \nabla_c M_{ab} + M_{cb} \nabla_a \chi^c + M_{ca} \nabla_b \chi^c = \phi^c M_{c(b} \nabla_a) \alpha. \quad (23)$$

Using Eq. (9), we find from the above equation

$$\begin{aligned} \chi^c \nabla_c M_{ab} &= \frac{1}{2} \phi^c M_{c(b} \nabla_a) \alpha - \frac{1}{2} (\nabla^c \alpha) M_{c(b} \phi_a) + \beta^{-1} (\nabla^c \beta) M_{c(b} \chi_a) - \beta^{-1} \chi^c M_{c(b} \nabla_a) \beta \\ &= H_{ab} \text{ (say)}. \end{aligned} \quad (24)$$

Using this and the fact that $M = 0$ we now find from Eq. (18),

$$\int_{\mathcal{H}} M_{ab} \nabla_c M_{ab} d\mathcal{H}^c + \int [dX] [-\beta^{-2} H_{ab}^2 - (D_c M_{ab})(D^c M_{ab}) + m^2 M_{ab}^2] \text{ (no sum on } a, b), \quad (25)$$

where $[dX]$ is the full spacetime volume measure and the direction 'c' in the horizon integral directs along μ^a . By our choice μ^a coincides with χ^a on \mathcal{H} (Eq.s (10), (11)), so that the integrand in the horizon integral coincides with $M_{ab} H_{ab}$. Let us first set $a = 0, b = 1$ in the above integrals. Using the fact that $\nabla_a \alpha$ and $\nabla_a \beta$ are both orthogonal to χ^a and ϕ^a (Eq.s (7)), and four of the ten components of M_{ab} are already zero, we find from Eq. (24) that $H_{01} = 0$. Then Eq. (25) shows that $M_{01} = 0$ throughout. Similarly we can show that all the other components of M_{ab} vanish also.

Thus all the six components of M_{ab} vanish identically in the region between the black hole and cosmological horizon. This is the expected classical no hair result for this field. For asymptotically flat or anti-de Sitter spacetimes ($\Lambda \leq 0$), the boundary integral at the cosmological horizon is replaced by an integral at spacelike infinity. By imposing sufficiently rapid fall-off condition on the matter field, we can make the integral vanishing and the desired no hair result follows.

It was shown in [23, 24] for static spherically symmetric spacetimes that although classical spin-2 hair is ruled out, quantum hair is not. The idea is the following. A Stückelberg field A_b was introduced to write M_{ab} as

$$M_{ab} = \widehat{M}_{ab} + \nabla_a A_b + \nabla_b A_a. \quad (26)$$

Then M_{ab} is invariant under the local gauge transformations : $A_b \rightarrow A_b - \zeta_b$, $\widehat{M}_{ab} \rightarrow \widehat{M}_{ab} + \nabla_{(a}\zeta_{b)}$. Since $M_{ab} = 0$, one has $\widehat{M}_{ab} = -(\nabla_a A_b + \nabla_b A_a)$. Then a magnetic monopole solution for $F_{ab} = \nabla_{[a}A_{b]}$ was constructed and it was shown that the magnetic charge can be detected via a stringy generalization of the Bohm-Aharonov effect in the asymptotic region. In this work we have shown that M_{ab} vanishes also for general stationary axisymmetric spacetimes. Following this, we can break M_{ab} into two gauge fields, from one of which we can construct a magnetic monopole solution. It is clear that the solution will not be spherically symmetric in this case. However, if the black hole is small compared to the cosmological horizon size, spacetime will be spherically symmetric at large distance from the black hole, and the solution will asymptotically reach the usual spherically symmetric monopole solution. Accordingly, we can detect in this region a magnetic charge of the black hole. It remains as an interesting task to construct explicitly such monopole solutions, for example for the Kerr-de Sitter spacetime.

4 Massive spin- $\frac{1}{2}$ field

Let us now consider the case of a massive spin- $\frac{1}{2}$ field. The detailed formalism of such fields in curved spacetime can be found in e.g. [34, 36, 37]. The Lagrangian is given by

$$\mathcal{L} = \frac{i}{2} [\bar{\Psi} \gamma^a \nabla_a \Psi - (\nabla_a \bar{\Psi}) \gamma^a \Psi] - m \bar{\Psi} \Psi, \quad (27)$$

where Ψ is a 4-component spinor. The covariantly constant matrices γ^a 's can be expanded in an orthonormal basis $\gamma^a = e^a_{(b)} \gamma^{(b)}$. Using the well known anticommutation relation, $[\gamma^{(a)}, \gamma^{(b)}]_+ = 2\eta^{(a)(b)} \mathbf{I}$, where \mathbf{I} is the 4×4 identity matrix, we find

$$[\gamma^a, \gamma^b]_+ = 2g^{ab} \mathbf{I}. \quad (28)$$

The adjoint spinor $\bar{\Psi}$ is defined as $\bar{\Psi} = \Psi^\dagger \gamma^{(0)}$. The matrix $\gamma^{(0)}$ is Hermitian whereas $\gamma^{(i)}$, $i = 1, 2, 3$, are anti-Hermitian. The spin covariant derivative '∇' in Eq. (27) is defined as

$$\nabla_a \Psi = \partial_a \Psi + \frac{1}{8} \omega_{a(b)(c)} [\gamma^{(b)}, \gamma^{(c)}] \Psi, \quad \nabla_a \bar{\Psi} = \partial_a \bar{\Psi} - \frac{\bar{\Psi}}{8} \omega_{a(b)(c)} [\gamma^{(b)}, \gamma^{(c)}], \quad (29)$$

where $\omega_{a(b)(c)}$ are the Ricci rotation coefficients given by $\omega_{a(b)(c)} = e^d_{(b)} \nabla_a e_{(c)d}$. It is easy to show using Eq. (29) that [34, 36, 37],

$$[\nabla_a, \nabla_b] \Psi = -\frac{1}{8} R_{ab(c)(d)} [\gamma^{(c)}, \gamma^{(d)}] \Psi = -\frac{1}{8} R_{abcd} [\gamma^c, \gamma^d] \Psi, \quad (30)$$

using the fact that contraction is independent of basis. The equations of motion are given by

$$i\gamma^a \nabla_a \Psi - m\Psi = 0, \quad i(\nabla_a \bar{\Psi}) \gamma^a + m\bar{\Psi} = 0. \quad (31)$$

We consider the conserved current 1-form J_a ,

$$J_a = \bar{\Psi} \gamma_a \Psi : \nabla_a J^a = 0, \quad (32)$$

by Eq.s (31). Let us define a 2-form S_{ab} ,

$$S_{ab} := \nabla_{[a} J_{b]}, \quad (33)$$

so that

$$\nabla^a S_{ab} = \nabla_a \nabla^a (\bar{\Psi} \gamma_b \Psi) - R_b^a (\bar{\Psi} \gamma_a \Psi). \quad (34)$$

Since we ignore backreaction in this case, we have $R_{ab} = \Lambda g_{ab}$. Then setting $b = 0$ above and noting $\gamma_0 = \beta\gamma_{(0)}$ in our orthogonal basis, we find the following

$$\nabla^a S_{a0} = \nabla_a \nabla^a (\beta \Psi^\dagger \Psi) - \Lambda \beta \Psi^\dagger \Psi. \quad (35)$$

Integrating the above equation using the full spacetime volume element $[dX]$ and converting the total divergences into surface integrals on \mathcal{H} we get

$$\int_{\mathcal{H}} S_{a0} d\mathcal{H}^a - \int_{\mathcal{H}} \nabla_a (\beta \Psi^\dagger \Psi) d\mathcal{H}^a + \int [dX] \Lambda \beta \Psi^\dagger \Psi = 0, \quad (36)$$

where the unit normal ‘ a ’ as before directs along μ^a . It is clear that the measures on \mathcal{H} are non-divergent. Since S_{ab} is antisymmetric in its indices and by our choice μ_a coincides with χ_a on \mathcal{H} (Eq.s (10), (11)), the first integral vanishes in Eq. (36). Let us now evaluate the second boundary integral. Eq. (17) implies $\mathcal{L}_\xi J_a = 0 = \mathcal{L}_\phi J_a$, which gives

$$\chi^a \nabla_a (\bar{\Psi} \gamma_b \Psi) = \frac{(\bar{\Psi} \gamma^a \Psi)}{2\beta^2} [\chi_a \nabla_b \beta^2 - \chi_b \nabla_a \beta^2] + \frac{1}{2} (\bar{\Psi} \gamma^a \Psi) [\phi_a \nabla_b \alpha - \phi_b \nabla_a \alpha], \quad (37)$$

where we have used Eq. (9). Setting $b = 0$ and using Eq.s (7) we get

$$\chi^a \nabla_a (\beta \Psi^\dagger \Psi) = -\frac{(\bar{\Psi} \gamma_a \Psi \nabla^a \beta^2)}{2\beta^2} \chi_0. \quad (38)$$

Since μ^a coincides with χ^a on \mathcal{H} , the second integrand in Eq. (36) is given by the above expression. Then from the fact that $\bar{\Psi} \gamma_0 \Psi = \beta \Psi^\dagger \Psi$, it is clear that the above quantity is $\mathcal{O}(\beta)$ when evaluated on \mathcal{H} . This implies the second integral in Eq. (36) also vanishes. This shows that $\Psi = 0$ throughout. For $\Lambda < 0$ the outer boundary is infinity and suitable fall-off condition for the massive field recovers the no hair result.

The above simple proof is however not valid for $\Lambda = 0$. Unfortunately we have been able to do the proof for such spacetimes only under stronger assumption than the above. It is the following.

We multiply the first of Eq.s (31) by $i\gamma^b \nabla_b$ and use Eq.s (28), (30) to get

$$\nabla_a \nabla^a \Psi - \frac{1}{32} R_{abcd} [\gamma^a, \gamma^b] [\gamma^c, \gamma^d] \Psi + m^2 \Psi = 0. \quad (39)$$

We shall now simplify the second term. Denoting $[\gamma^a, \gamma^b]$ by $\tilde{\sigma}^{ab}$, we compute

$$\tilde{\sigma}^{a[b} \tilde{\sigma}^{cd]} = 2 [\tilde{\sigma}^{ab} \tilde{\sigma}^{cd} + \tilde{\sigma}^{ac} \tilde{\sigma}^{db} + \tilde{\sigma}^{ad} \tilde{\sigma}^{bc}]. \quad (40)$$

Contracting both sides by R_{abcd} , recalling the identity $R_{a[bcd]} = 0$, and the symmetries of the Riemann tensor we find

$$R_{abcd} \tilde{\sigma}^{ab} \tilde{\sigma}^{cd} = 2 R_{abcd} \tilde{\sigma}^{ac} \tilde{\sigma}^{bd}. \quad (41)$$

Using the anticommutation relations for the γ ’s we find from the above

$$R_{abcd} \tilde{\sigma}^{ab} \tilde{\sigma}^{cd} = 8 [R_{abcd} \gamma^a \gamma^c \gamma^b \gamma^d - R]. \quad (42)$$

The first term can be written as

$$R_{abcd} \gamma^a \gamma^c \gamma^b \gamma^d = R_{abcd} \gamma^a (2g^{bc} - \gamma^b \gamma^c) \gamma^d = -2R - R_{abcd} \gamma^a \gamma^b \gamma^c \gamma^d = -2R - \frac{1}{4} R_{abcd} \tilde{\sigma}^{ab} \tilde{\sigma}^{cd}, \quad (43)$$

using the fact that $R_{abcd} (\gamma^a \gamma^b - \gamma^b \gamma^a) \gamma^c \gamma^d = R_{abcd} (2\gamma^a \gamma^b - 2g^{ab}) \gamma^c \gamma^d = 2R_{abcd} \gamma^a \gamma^b \gamma^c \gamma^d$. Putting in all this we have from Eq. (42)

$$R_{abcd} \tilde{\sigma}^{ab} \tilde{\sigma}^{cd} = -8R. \quad (44)$$

Thus Eq. (39) now simplifies to [37],

$$\nabla_a \nabla^a \Psi + \left(\frac{R}{4} + m^2 \right) \Psi = 0. \quad (45)$$

It can be seen from Eq. (29) that $(\nabla_a \Psi)^\dagger = \nabla_a \Psi^\dagger$, so that Ψ^\dagger satisfies the same equation as above. From Einstein's equations we get

$$R = 4(\Lambda - 2\pi T). \quad (46)$$

Multiplying Eq. (45) by Ψ^\dagger and using the projector $\pi_a{}^b$ defined in Eq. (16) we now compute

$$\begin{aligned} \overline{D}_a \overline{D}^a (\Psi^\dagger \Psi) &:= \pi^{ab} \nabla_a \nabla_b (\Psi^\dagger \Psi) = 2 (\nabla_a \Psi^\dagger) (\nabla_a \Psi) - 2 \left(m^2 + \frac{R}{4} \right) \Psi^\dagger \Psi \\ &\quad + [f^{-2} \phi^a \nabla_a (\phi^b \nabla_b (\Psi^\dagger \Psi)) - \beta^{-2} \chi^a \nabla_a (\chi^b \nabla_b (\Psi^\dagger \Psi))] \\ &\quad - [f^{-1} (\overline{D}_a f) (\overline{D}_a (\Psi^\dagger \Psi)) + \beta^{-1} (\overline{D}_a \beta) (\overline{D}_a (\Psi^\dagger \Psi))], \end{aligned} \quad (47)$$

where we have used equations of motion for Ψ and Ψ^\dagger and the fact that $\nabla_a \beta = D_a \beta = \overline{D}_a \beta$ and $\nabla_a f = D_a f = \overline{D}_a f$, which follow from Eq.s (7). The above can be rewritten as

$$\begin{aligned} \overline{D}_a \left[f \beta \overline{D}^a (\Psi^\dagger \Psi) \right] &= 2 f \beta (\overline{D}_a \Psi^\dagger) (\overline{D}_a \Psi) - 2 f \beta \left(m^2 + \frac{R}{4} \right) \Psi^\dagger \Psi \\ &\quad + f \beta [f^{-2} [\phi^a \nabla_a (\phi^b \nabla_b \Psi^\dagger) \Psi + \Psi^\dagger \phi^a \nabla_a (\phi^b \nabla_b \Psi)] \\ &\quad - \beta^{-2} [\chi^a \nabla_a (\chi^b \nabla_b \Psi^\dagger) \Psi + \Psi^\dagger \chi^a \nabla_a (\chi^b \nabla_b \Psi)]]]. \end{aligned} \quad (48)$$

Let us now simplify the last four terms of this equation using symmetry arguments. Our assumption in this case will be $\mathcal{L}_\xi \Psi = 0 = \mathcal{L}_\phi \Psi$, which is of course much stronger than the previous one made on the conserved current 1-form.

The definition of the Lie derivative of a spinor requires the notion of Lie derivative on a fiber bundle. We refer our reader to [38] for a detailed discussion on this including an exhaustive list of references. The Lie derivatives of a spinor Ψ and its adjoint $\overline{\Psi}$ along any Killing vector field X is given by

$$\mathcal{L}_X \Psi = X^a \nabla_a \Psi - \frac{1}{8} \nabla_{[a} X_{b]} \gamma^a \gamma^b \Psi, \quad \mathcal{L}_X \overline{\Psi} = X^a \nabla_a \overline{\Psi} + \frac{1}{8} \overline{\Psi} \nabla_{[a} X_{b]} \gamma^a \gamma^b. \quad (49)$$

It is easy to see that in a local coordinate system in which $X^a = (\partial_x)^a$, where x is the coordinate along X^a , the above formula reduces to the directional partial derivative along X^a . This is compatible with our common intuition about Lie derivatives. Thus for the customary dependence $e^{i(\omega t - m\phi)}$, the above conditions simply mean $\omega = 0 = m$. Such condition was used previously in [14] for spherically symmetric static spacetime.

Using Eq.s (49), (9) and (16) we have

$$\begin{aligned} \chi^a \nabla_a \Psi &= \frac{\beta^{-1}}{4} [\chi_b \nabla_a \beta - \chi_a \nabla_b \beta] \gamma^a \gamma^b \Psi + \frac{1}{4} \phi_a \nabla_b \alpha \gamma^a \gamma^b \Psi, \\ \phi^a \nabla_a \Psi &= \frac{f^{-1}}{4} [\phi_b \nabla_a f - \phi_a \nabla_b f] \gamma^a \gamma^b \Psi + \frac{f^2}{8\beta^2} [\chi_a \nabla_b \alpha - \chi_b \nabla_a \alpha] \gamma^a \gamma^b \Psi. \end{aligned} \quad (50)$$

The corresponding expressions for the derivatives of Ψ^\dagger can be found from the second of Eq.s (49) by multiplying it by $\gamma^{(0)}$ from right and using the anticommutation relations for the gamma matrices. We note that since $\nabla_a \beta$, $\nabla_a f$ and $\nabla_a \alpha$ are orthogonal to χ^a and ϕ^a , in contractions like

$\chi_a(\nabla_b\beta)\gamma^a\gamma^b$, $\phi_a(\nabla_b\alpha)\gamma^a\gamma^b$, the gamma matrices must anticommute. Using this and Eq. (28), we find from Eq.s (50) after a lengthy but straightforward computation,

$$\begin{aligned}\chi^a\nabla_a(\chi^b\nabla_b\Psi^\dagger)\Psi + \Psi^\dagger\chi^a\nabla_a(\chi^b\nabla_b\Psi) &= -\frac{1}{2}(\nabla_a\beta)(\nabla^a\beta)\Psi^\dagger\Psi + \frac{f^2}{8}(\nabla_a\alpha)(\nabla^a\alpha)\Psi^\dagger\Psi, \\ \phi^a\nabla_a(\phi^b\nabla_b\Psi^\dagger)\Psi + \Psi^\dagger\phi^a\nabla_a(\phi^b\nabla_b\Psi) &= \frac{1}{2}(\nabla_af)(\nabla^af)\Psi^\dagger\Psi - \frac{f^4}{8\beta^2}(\nabla_a\alpha)(\nabla^a\alpha)\Psi^\dagger\Psi.\end{aligned}\quad (51)$$

Substituting these into Eq. (48) we get

$$\begin{aligned}\overline{D}_a\left[f\beta\overline{D}^a(\Psi^\dagger\Psi)\right] &= 2f\beta(\overline{D}_a\Psi^\dagger)(\overline{D}_a\Psi) - 2f\beta\left(m^2 + \frac{R}{4}\right)\Psi^\dagger\Psi \\ &+ \frac{f\beta}{2}\left[\beta^{-2}(\nabla_a\beta)(\nabla^a\beta) + f^{-2}(\nabla_af)(\nabla^af) - \frac{f^2}{2\beta^2}(\nabla_a\alpha)(\nabla^a\alpha)\right]\Psi^\dagger\Psi,\end{aligned}\quad (52)$$

which we integrate to find

$$\begin{aligned}\int d\Sigma\beta\left[2(\overline{D}_a\Psi^\dagger)(\overline{D}_a\Psi) + \frac{1}{2}(\beta^{-2}(\nabla_a\beta)(\nabla^a\beta) + f^{-2}(\nabla_af)(\nabla^af) \right. \\ \left. - \frac{f^2}{2\beta^2}(\nabla_a\alpha)(\nabla^a\alpha) - 4\left(m^2 + \frac{R}{4}\right))\right]\Psi^\dagger\Psi = 0,\end{aligned}\quad (53)$$

where we have used the fact that $\int f d\overline{\Sigma} = \frac{1}{2\pi} \int d\Sigma$, since none of the integrand depends on the Killing parameter ϕ and by definition it ranges from 0 to 2π . All but the fourth and the last term in the above equation are negative definite. The fourth term is positive and can naively be interpreted as the repulsive effect of the spacetime rotation on matter field. If we set $\alpha = 0$ in Eq. (53), we recover the static spacetime equation.

We shall now examine whether the term due to rotation can dominate the integral (53). To do this, let us consider the Killing identity for ϕ_a ,

$$\nabla_b\nabla^b\phi_a = -R_a{}^b\phi_b, \quad (54)$$

which we contract by ϕ^a and use Eq. (16) to get

$$\nabla_a\nabla^a f = f^{-1}(\nabla_af)(\nabla^af) - \frac{f^3}{2\beta^2}(\nabla_a\alpha)(\nabla^a\alpha) - f^{-1}R_{ab}\phi^a\phi^b. \quad (55)$$

We project this equation onto Σ using the techniques described earlier, use Einstein's equations without backreaction and multiply by $f^{-1}\Psi^\dagger\Psi$ to find

$$D_a\left[\beta f^{-1}\Psi^\dagger\Psi D^a f\right] = \beta\left[\left(-\frac{f^2}{2\beta^2}(\nabla_a\alpha)(\nabla^a\alpha) + \Lambda\right)\Psi^\dagger\Psi + f^{-1}(D_a(\Psi^\dagger\Psi))(D^a f)\right], \quad (56)$$

which we integrate between the two horizons. The boundary integrals go away and we combine the vanishing volume integral with Eq. (53) to get

$$\begin{aligned}\int d\Sigma\beta\left[(\overline{D}_a\Psi^\dagger)(\overline{D}_a\Psi) + \frac{1}{2}\left(\beta^{-2}(\nabla_a\beta)(\nabla^a\beta) + \frac{f^{-2}}{2}(\nabla_af)(\nabla^af) - 4\left(m^2 + \frac{5}{4}\Lambda\right)\right)\Psi^\dagger\Psi \right. \\ \left. + \left(\overline{D}_a\Psi - \frac{f^{-1}}{2}\Psi\overline{D}_a f\right)^\dagger\left(\overline{D}^a\Psi - \frac{f^{-1}}{2}\Psi\overline{D}^a f\right)\right] = 0,\end{aligned}\quad (57)$$

where we have used the fact that $\nabla_af = D_af = \overline{D}_af$ (Eq.s (7)). All the terms are negative definite now, which shows that $\Psi = 0$ throughout our region of interest, which is the desired no

hair result. This result clearly holds for $\Lambda = 0$ provided we impose suitable fall-off condition at spatial infinity. This also holds for an asymptotically anti-de Sitter spacetime if in addition to the fall-off condition, we assume that $m^2 \geq \frac{5}{4}|\Lambda|$, which means that the Compton wavelength of the spinor is small compared to the AdS length scale.

5 Summary

In this work we have proved no hair theorems for massive spin-2 and spin- $\frac{1}{2}$ fields for general stationary axisymmetric de Sitter black hole spacetimes. The existence of quantum hair for the spin-2 field was also discussed. Since spinors do not satisfy any classical energy condition, the no spinor hair could only be proved upon imposition of weakness condition. The backreaction of spinors should involve renormalization of the energy-momentum tensor, which seems an interesting problem in stationary axisymmetric spacetime. It will be interesting to investigate the situation when the spinor gets coupled to a gauge field, a Maxwell field for example.

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